ISRAEL JOURNAL OF MATHEMATICS, Vol. 30, No. 3, 1978

IF A FINITE EXTENSION OF A BERNOULLI SHIFT HAS NO FINITE ROTATION FACTORS, IT IS BERNOULLI

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ABSTRACT

We show here that a finite extension of a Bernoulli shift either has a finite rotation factor or is Bernoulli. The proof lifts to this more general case the "nesting" technique we used previously to prove this for two point extensions.

In an earlier paper, *If a two-point extension of a Bernoulli shift has an ergodic square, then it is Bernoulli*, we developed a technique for investigating the behavior of two point extensions of a v.w.b. distribution. We here extend that result to finite extensions. Our attack is essentially identical to the two-point case. As the arguments get a good deal more involved, it was convenient to separate off, in the two-point case, those results concerning Bernoulli shifts which are completely general, and to provide a simpler case to familiarize both the reader and the author with the technique. The reader is expected to be familiar with this earlier argument, as it will be quoted from and applied very often.

A finite extension, $\hat{T}$, of a Bernoulli shift is a skew product of a Bernoulli action $(T, \Omega, \mu, \mathscr{F})$ with the actions of S_n, the symmetric group, on n-points. It is defined by $\hat{T}(\omega, i) = (T(\omega), g_\omega(i))$ where $g_\omega ; \Omega \to S_n$ is a measurable map. This is equivalent to saying $\hat{T}$ has an invariant Bernoulli factor with n-point fibers.

† This work was supported by the Miller Institute for Basic Research.

Received March 3, 1977

Here are some simple examples. For any $\alpha \in S_n$, if $g \equiv \alpha$, then $\hat{T}$ is not weakly mixing and in fact, for some k, $\hat{T}^k$ is nonergodic. On the other hand, if we define $g_\omega = \alpha_i$ when $\omega \in S_i$, $i = 1, \cdots, n!$, where $\{S_i\}$ is an independent generator for T, it is easy to check that $\hat{T}$ is mixing Markov on $\{S_i\} \vee \{1, \cdots, n\}$ and hence is isomorphic to a Bernoulli process. A simple lemma quickly limits us nearly to these.

LEMMA 1. *If $\hat{T}$ is an n-point extension of a Bernoulli shift and has no periodic factors, then $\hat{T}$ is a K-automorphism.*

PROOF. Suppose $\hat{T}$ is not K. Let S be a set in the Pinsker algebra $\pi(\hat{T})$. This S must be independent of the Bernoulli factor of $\hat{T}$. Thus S must intersect a.e. n-point fiber in some constant $k(s)$ points. As $\pi(\hat{T})$ is a σ-algebra, it thus must be atomic. Hence $\pi(\hat{T})$ must be a periodic factor. ∎

This lemma is a simple extension of Parry's result [2]. Our work now is to change K in the above to Bernoulli.

To begin, let T be a fixed n-point extension of a Bernoulli shift. For our purposes, a partition P of a space Y will be a measurable function from Y to some finite or countably infinite space of labels. The "sets in P" will be the inverse images of points in the label set. With this proviso, let P be a generating partition for the Bernoulli factor such that the sets on which g is constant are unions of sets in P. Let D be the partition of $\Omega \times \{1, \cdots, n\}$ such that $D((\omega, i)) = D_i$, i.e. the partition into sets of constant second coordinate, and let $\bar{P} = P \vee D$, $P \vee D(\omega, i) = (P(\omega, i), D(\omega, i))$. Now for any $\bar{\omega} \in \Omega \times \{1, \cdots, n\}$, if we know the P-names, $\cdots P(T^{-1}(\bar{\omega})) \cdots P(\bar{\omega}) \cdots P(T^i(\bar{\omega})) \cdots$, of $\bar{\omega}$, and its "color" $D(T^i(\bar{\omega}))$ at one index i, the P-name explicitly forces the rest of the colored name.

To each label in the range of P, the action g assigns an element of S_n. This map, though, may not be onto all of S_n. If not, extend the set of labels for P, called η, so that this assignment is onto. For $\beta \in \eta$, let $\Pi(\beta)$ be the element in S_n assigned to β. Now for any partition B with label space η, we can define $\Pi(\beta)$; $Y \to S_n$ by $\Pi(B)(y) = \Pi(B(y))$. For any partition B with label space η and any C with label space $\Gamma = \{D_1, \cdots, D_n\}$ we can define a partition $\Pi(B)(C)$ with label space Γ by $\Pi(B)(C)(y) = D_{\Pi(B)(y)(l)}$ where $C(y) = D_l$. Notice $D_{i+1} = \Pi(P_i)(D_i)$.

To any sequence of partitions $\bigvee_{i=1}^n B_i$ where each B_i is labeled by η, a "coloring" will be a sequence of partitions $\bigvee_{i=1}^n C_i$, C_i labeled by Γ where $C_{i+1} = \Pi(B_i)(C_i)$. Such a sequence $\bigvee_{i=1}^m B_i \vee C_i$ will be called a "colored sequence" of partitions. Thus $\bigvee_{i=1}^n T^i(\bar{P})$ is a colored sequence of partitions.

From now on, partitions named with B's or P's will have label space η, and C's or D's will have label space Γ.

We recall now a number of notions from [3]. A "copy" of $\bigvee_{i=1}^m B_i$ is a sequence of partitions, $\bigvee_{i=1}^m \bar{B}_i$ of $[0,1]$ with $\mathrm{dist}(\bigvee_{i=1}^m B_i) = \mathrm{dist}(\bigvee_{i=1}^m \bar{B}_i)$. A "joining" of $\bigvee_{i=1}^m B_i$ and $\bigvee_{i=1}^m B'_i$ is a copy of each, $\bigvee_{i=1}^m \bar{B}_i \vee \bar{B}'_i$. We abbreviate $(1/m)\sum_{i=1}^m d(\bar{B}_i, \bar{B}'_i)$ as $d_m(\bigvee_{i=1}^m \bar{B}_i, \bigvee_{i=1}^m \bar{B}'_i)$. We say two colored sequences, $\bigvee_{i=1}^m B_i \vee C_i$ and $\bigvee_{i=1}^m B'_i \vee C'_i$ are "(ε, a)-close" if there exists a joining so that

$$d_m\left(\bigvee_{i=1}^m \bar{B}_i, \bigvee_{i=1}^m \bar{B}'_i\right) < \varepsilon$$

and

$$d_m\left(\bigvee_{i=1}^m \bar{C}_i, \bigvee_{i=1}^m \bar{C}'_i\right) < a.$$

The opposite notion is $\bigvee_{i=1}^m B_i \vee C_i$ and $\bigvee_{i=1}^m B'_i \vee C'_i$ are "(ε, a)-rigidly apart" if for any joining with

$$d_m\left(\bigvee_{i=1}^m \bar{B}_i, \bigvee_{i=1}^m \bar{B}'_i\right) < \varepsilon,$$

we must have

$$d_m\left(\bigvee_{i=1}^m \bar{C}_i, \bigvee_{i=1}^m \bar{C}'_i\right) > 1 - a.$$

In the two-point case we knew that $\bigvee_{i=1}^m B_i \vee C_i$ and $\bigvee_{i=1}^m B'_i \vee C'_i$ were (ε, a)-rigidly apart iff $\bigvee_{i=1}^m B_i \vee C_i$ and $\bigvee_{i=1}^m B_i \vee C_i^f$ were (ε, a)-rigidly close. In the general n-point extension we have no notion to replace f by. Hence we must work with both closeness and apartness.

As before, our argument hinges on the notion of nesting. We will use it in a number of contexts, so we will develop it here in a degree of generality. First, a bit more structure. The "blocking" of a joining $\bigvee_{i=1}^m \bar{B}_i \vee \bar{B}'_i$ will be $\{F_k, \{S(k,g)\}\}$ where $\{F_k\}$ is a partition of $[0,1]$ into subsets, first by what subsequence of indices the two names of a point differ, i.e. for $\omega \in F_k$, $\bar{B}_i(\omega) \neq \bar{B}'_i(\omega)$ precisely when $i = i(1,k)$, $i(2,k), \cdots$, second we refine this by $\bigvee_l \bar{B}_{i(l,k)} \vee \bar{C}_{i(l,k)} \vee \bar{B}'_{i(l,k)} \vee \bar{C}'_{i(l,k)}$, and third we partition by what element of S_n,

$$\Pi(\bar{B}_1/F_k) \circ \Pi(\bar{B}_2/F_k) \circ \cdots \circ \Pi(\bar{B}_{i(k,l)}/F_k) = g(k,l)$$

and

$$\Pi(\bar{B}'_1/F_k) \circ \Pi(\bar{B}'_2/F_k) \circ \cdots \circ \Pi(\bar{B}'_{i(k,l)}/F_k) = g'(k,l)$$

are. Finally $S(k,g)$ is, for each k, the partition into at most $n!$ sets of $\{1\cdots m\}$ where

$$S(k,g)=\{i(k,l)+t \mid g(k,l)^{-1}g(k,l)=g,\ 1\leqq t\leqq i(k,l+1)-i(k,l)\}.$$

A "monochrome" coloring, $\bigvee_{i=1}^{m} C_i$, for $\bigvee_{i=1}^{m} B_i$ is one for which C_i is $\bigvee_{i=1}^{m} B_i$ measurable, i.e. any uncolored name has exactly one color sequence.

Now we are ready to define what it means to nest. Let $\bigvee_{i=1}^{m_j} B_{i,j} \vee C_{i,j}$ and $\bigvee_{i=1}^{m_j} B'_{i,j} \vee C'_{i,j}$ be two sequences of distributions where $m_j \underset{j}{\rightarrow} \infty$ and

$$\bar{d}_{m_j}\left(\bigvee_{i=1}^{m_j} B_{i,j}, \bigvee_{i=1}^{m_j} B'_{i,j}\right)\underset{j}{\rightarrow} 0.$$

We say these two "nest to $\bar{\varepsilon}$ with rate A, $\delta(\quad)$" if the following holds. For any $\varepsilon>0$, there is a $J(\varepsilon)$. Take for any $j>J(\varepsilon)$ a joining

$$\bigvee_{i=1}^{m_j} \bar{B}_{i,j} \vee \bar{C}_{i,j} \vee \bar{B}'_{i,j} \vee \bar{C}'_{i,j}$$

with

$$d_{m_j}\left(\bigvee_{i=1}^{m_j} \bar{B}_{i,j}, \bigvee_{i=1}^{m_j} \bar{B}'_{i,j}\right)<\delta(\varepsilon).$$

Let $\{F_k,\{S(k,g)\}\}$ be its blocking. Let $\mathcal{S}(k)=\{i \mid \bar{C}_{i,j}/F_k=\bar{C}'_{i,j}/F_k\}$. This is precisely the places where the colors agree and is a union of $S(k,g)$'s. Finally, (i) for at least $1-\bar{\varepsilon}/2$ of the F_k, $\operatorname{card}(\mathcal{S}(k))\geqq(1-\bar{\varepsilon}/2)n$ or (ii) for at least A in measure of the F_k with $\operatorname{card}(\mathcal{S}(k))<(1-\varepsilon/2)n$, for at least A in density of the $S(k,g)\not\subset\mathcal{S}(k)$, for at least A in measure of the atoms E of

$$\bigvee_{i\in S(k,g)} \bar{B}_{i,j} \vee \bar{C}_{i,j} \vee \bar{B}'_{i,j} \vee \bar{C}'_{i,j},$$

$$\bigvee_{i\in S(k,g)} \bar{B}_{i,j} \vee \bar{C}_{i,j}/F_k \cap E \quad \text{and} \quad \bigvee_{i\in S(k,g)} \bar{B}'_{i,j} \vee \bar{C}'_{i,j}/F_k \cap E$$

are $(\varepsilon, 1-A)$-close.

LEMMA 2. *Given any* $\bigvee_{i=1}^{m_j} B_{i,j} \vee C_{i,j}$ *and* $\bigvee_{i=1}^{m_j} B'_{i,j} \vee C'_{i,j}$ *which nest at rate* A, $\delta(\quad)$, *then*

$$\lim_{j\to\infty} \bar{d}_{m_j}\left(\bigvee_{i=1}^{m_j} B_{i,j} \vee C_{i,j}, \bigvee_{i=1}^{m_j} B'_{i,j} \vee C'_{i,j}\right)=0.$$

PROOF. Call a joining with

$$\bar{d}_{m_j}\left(\bigvee_{i=1}^{m_j} \bar{B}_{i,j}, \bigvee_{i=1}^{m_j} \bar{B}'_{i,j}\right) < \varepsilon'$$

an "ε'-joining". Let

$$f(J, \varepsilon') = \sup_{j>J}\left(\inf_{\varepsilon'\text{-joinings}}\left(\bar{d}_{m_j}\left(\bigvee_{i=1}^{m_j} \bar{C}_{i,j}, \bigvee_{i=1}^{m_j} \bar{C}'_{i,j}\right)\right)\right).$$

Fix ε' and take $\varepsilon < \delta(\varepsilon')$, $j > J(\varepsilon)$. Consider an ε-joining of $\bigvee_{i=1}^{m_j} B_{i,j} \vee C_{i,j}$ and $\bigvee_{i=1}^{m_j} B'_{i,j} \vee C'_{i,j}$. If (i) holds we are done. Otherwise in its blocking, ignore the F_k with card $(\mathcal{S}(k)) \geqq (1-\bar{\varepsilon}/2)n$ and the further fraction $1-A$ of the F_k about which nesting says nothing. Of the $S(k,g) \not\subset \mathcal{S}(k)$ we can select one, $S(k,g_k)$. card$(S(k,g_k)) > m_jA/(n!)^2$, where nesting says we can condition on everything outside $S(k,g_k)$ and improve the joining. Do so, and we conclude if $f(J(\varepsilon),\varepsilon) > \bar{\varepsilon}$, then

$$\text{(i)}\quad f(J(\varepsilon), \varepsilon+\varepsilon') \leqq \varepsilon' + f(J(\varepsilon),\varepsilon) - (f(J(\varepsilon),\varepsilon) - 2\varepsilon')\left(\frac{A^4}{2n^2(n!)^2}(1-\varepsilon')\right)$$

$$< 3\varepsilon' + f(J(\varepsilon),\varepsilon)\left(1 - \frac{A^4}{4n^2(n!)^2}\right)$$

whenever $\varepsilon \leqq \delta(\varepsilon')$. Now choose $\varepsilon_1, \cdots, \varepsilon_k$, where $\varepsilon_{l-1} = \inf(\delta(\varepsilon_l)/2, \varepsilon_l/2)$. Then $\varepsilon_1 + \cdots + \varepsilon_{l-1} \leqq \delta(\varepsilon_l)$. Iterating (i) k times, either at some step we are finished already as $f(J(\varepsilon_1), \varepsilon_1 + \cdots + \varepsilon_t) \leqq \bar{\varepsilon}$, or

$$f(J(\varepsilon_1), \varepsilon_1 + \cdots + \varepsilon_k) \leqq 3(\varepsilon_2 + \cdots + \varepsilon_k) + f(J(\varepsilon_1), \varepsilon_1)\left(1 - \frac{A^4}{4n^2(n!)^2}\right)^k$$

$$\leqq 3(\varepsilon_2 + \cdots + \varepsilon_k) + \left(1 - \frac{A^4}{4n^2(n!)^2}\right)^k.$$

For any ε we can choose k so that $\varepsilon_1 + \cdots \varepsilon_k < \varepsilon/5$ and $(1 - A^4/4n^2(n!)^2)^k < \varepsilon/5$ and we are done. ■

We now begin to investigate something which, at first, may not appear that natural, but as we saw in [3], it is the proper context in which to handle the failure of nesting to occur.

Consider, then a doubly indexed array of distributions

$$\bigvee_{i=1}^{m(j,t)} B_i(j,t), \quad (j,t) \in (\mathbf{Z}^+)^2,$$

where for fixed t, $m(j,t) \underset{j}{\to} \infty$ and

$$\lim_{j\to\infty}\left(\bar d_{m(j,t)}\left(\bigvee_{i=1}^{m(j,t)} B_i(j,t), \bigvee_{i=1}^{m(j,t)} T^i(P)\right)\right)=0.$$

To each element in this array assign n different monochrome colorings $\bigvee_{i=1}^{m(j,t)} C_i^l(j,t)$, $l=1,\cdots,n$, so that for $l\neq l'$,

$$d(C_1^l(j,t), C_1^{l'}(j,t))=1,$$

i.e. no two ever agree anywhere. We cannot have more than n such. Call this a "disjointly colored array," and the n colorings a "disjoint coloring."

Suppose we have two disjointly colored arrays with $m(j,t)=m'(j,t)$, and a collection L of ordered pairs $(l,l')\in\{1,\cdots,n\}^2$, so that when $(l,l')\in L$,

$$\bigvee_{i=1}^{m(j,t)} B_i(j,t)\vee C_i^l(j,t) \quad\text{and}\quad \bigvee_{i=1}^{m(j,t)} B_i'(j,t)\vee C_i'^{l'}(j,t)$$

are $(\varepsilon(t),1/t)$-rigidly apart, for some $\varepsilon(t)>0$. What interests us here is maximizing the size of L. Certainly a maximum exists, as $\operatorname{card}(L)\leqq n^2$, but we want to approach its construction systematically, and understand how it relates to nesting. We begin by introducing a scheme for modifying the disjointly colored arrays to increase the number of pairs in L.

Corollary 3. *Let*

$$\bigvee_{i=1}^{m(j,t)} B_i(j,t)\vee \bigvee_{l=1}^{n} C_i^l(m,t)$$

and

$$\bigvee_{i=1}^{m(j,t)} B_i'(j,t)\vee \bigvee_{l=1}^{n} C_i'^l(j,t)$$

be two disjointly colored arrays with pairs, L, of colors $(\varepsilon(t),1/t)$-rigidly apart. Let $(l,l')\notin L$. Then either

i) *for any $\varepsilon_1>0$ there is a T so that for $t\geqq T$ and any $\varepsilon_2>0$ there is a J so that for $j>J$,*

$$\bigvee_{i=1}^{m(j,t)} B_i(j,t)\vee C_i^l(j,t) \quad\text{and}\quad \bigvee_{i=1}^{m(j,t)} B_i'(j,t)\vee C_i'^{l'}(j,t)$$

are $(\varepsilon_2,\varepsilon_1)$-close, or

ii) *for any $A>0$ there is an $\bar\varepsilon>0$ and an infinite set $T(A)$ so that for $t\in T(A)$ there is an $\hat\varepsilon(t,A)>0$ so that for any $\delta>0$, there is an infinite set $J(A,t,\delta)$ so that for $j\in J(A,t,\delta)$, there is a δ-match*

$$\bigvee_{i=1}^{m(j,t)} \bar B_i(j,t)\vee \bar C_i^l(j,t)\vee \bar B_i'(j,t)\vee \bar C_i'^{l'}(j,t)$$

so that in its blocking, for more than $(1-A)\bar{\varepsilon}$ *of the* F_k, *for more than* $(1-A)\bar{\varepsilon}$ *of the* $S(k,g)$, *for more than* $(1-A)$ *of the atoms* E *in* $\bigvee_{i\in S^c(k,g)} \bar{B}_i(j,t) \vee \bar{C}_i^l(j,t) \vee \bar{B}_i(j,t) \vee \bar{C}_i'^{l'}(j,t)$,

$$\bigvee_{i\in S(k,g)} \bar{B}_i(j,t) \vee \bar{C}_{i,j}^l(j,t)/E\cap F_k \quad \textit{and} \quad \bigvee_{i\in S(k,g)} \bar{B}'_{i,j} \vee \bar{C}_{i,j}'^{l'}(j,t)/F_k\cap E$$

are $(\hat{\varepsilon}(t,A),A)$*-rigidly apart.*

Proof. If i) is false, then for any $A>0$, there is an $\bar{\varepsilon}>0$ and infinite set $T(A)$ so that for $t\in T(A)$, there is no $\delta(\ )$ for which the sequence in j nests to $\bar{\varepsilon}$ with rate A, $\delta(\ )$. Hence for some $\hat{\varepsilon}(t,A)$, no matter how small δ is chosen, it must fail as a choice for $\delta(\hat{\varepsilon}(t,A))$, for infinitely many j, in the manner ii) describes. ∎

In case ii), we have in

$$\bigvee_{i\in S(k,g)} \bar{B}_i(j,t) \cup \bar{C}_{i,j}^l(j,t)/E\cap F_k \quad \text{and} \quad \bigvee_{i\in S(k,g)} \bar{B}'_{i,j} \vee \bar{C}_{i,j}'^{l'}(j,t)/E\cap F_k$$

two distributions to try to use as new terms in a set of pairs rigidly apart. We have three difficulties to handle. These are not colored distributions. We may lose the other rigidly apart pairs. These must be chosen to form disjointly colored arrays.

To handle the first problem, when we reindex these two in consecutive order, we want to modify the uncolored names when necessary so that the colored sequences are colorings for it. Hence, let $S(k,g)=\{j(1),j(2),\cdots,j(\operatorname{card} S(k,g))\}$ and for $j(l)\neq i(k,l')$ for any l', let

$$\bar{B}_l(k,g,E)=\bar{B}_{j(l)}/E\cap F_k=\bar{B}'_{j(l)}/E\cap F_k.$$

If $j(l)=i(k,l')$ then $j(l+1)=i(k,l'')+1$ for some l'', and for any $\omega\in E\cap F_k$,

$$\begin{aligned}
&\Pi(\bar{B}^{(\omega)}_{i(k,l')}\circ\cdots\circ\Pi(\bar{B}^{(\omega)}_{i(k,l'')})\\
&\quad=(\Pi(\bar{B}^{(\omega)}_{i(k,l-1)+1}\circ\cdots\circ\Pi(\bar{B}^{(\omega)}_{i(k,l'')-1}))^{-1}\circ g(k,l'-1)^{-1}g(k,l'')\\
&\quad=\Pi(\bar{B}'^{(\omega)}_{i(k,l'-1)+1})\circ\cdots\circ\Pi(\bar{B}'^{(\omega)}_{i(k,l'')-1}))^{-1}\circ g'(k,l'-1)^{-1}g'(k,l'').
\end{aligned}$$

Let $\bar{B}_l(k,g,E)$ be some partition so that $\Pi(\bar{B}_l(k,g,E))$ is this one. Now let

$$\bar{C}_l(k,g,E)\vee\bar{C}'_l(k,g,E)=\bar{C}_{j(l)}\vee\bar{C}'_{j(l)}/E\cap F_k.$$

It follows that both of these are monochrome colorings for $\bigvee_{i=1}^{\operatorname{card}(S(k,g))}\bar{B}_i(k,g,E)$. Call this the "standard modification" of these two conditional distributions. More generally, whenever we consider the distribution across some subset of indices, conditioned on a single name across the remaining indices, we can modify the uncolored conditional distribution only at those indices i where $i+1$ is not in the subset, so that, when collapsed down, we get a colored distribution.

Our next lemma settles the second problem.

LEMMA 4. *If for two disjointly colored arrays, we have an $\varepsilon(t)>0$, and (l,l') so that*

$$\bigvee_{i=1}^{m(j,t)} B_i(j,t) \vee C_i^l(j,t) \quad \text{and} \quad \bigvee_{i=1}^{m(j,t)} B_i'(j,t) \vee C_i'^{l'}(j,t)$$

are $(\varepsilon(t), 1/t)$-rigidly apart, then for any $\delta < \varepsilon(t)/2$ and any δ-match of these two, for all but $\sqrt[4]{1/t}$ of the F_k, for all but $\sqrt[4]{1/t}$ of the $S(k,g)$, for all but $\sqrt[4]{1/t}$ of the atoms E in

$$\bigvee_{i\in S(k,g)} \bar{B}_{i,j}(j,t) \vee \bar{C}_{i,j}^l(j,t) \vee \bar{B}_{i,j}'(j,t) \vee \bar{C}_{i,j}'^{l'}(j,t),$$

$$\bigvee_{i\in S(k,g)} \bar{B}_i(j,t) \vee \bar{C}_{i,j}^l(j,t)/F_k \cap E \quad \text{and} \quad \bigvee_{i\in S(k,g)} \bar{B}_i'(j,t) \vee \bar{C}_{i,j}'^{l'}(j,t)/F_k \cap E$$

are still $(\varepsilon(t)/2, n^2(n!)^2\sqrt[4]{1/t})$-rigidly apart.

PROOF. Otherwise, starting at this δ-match, we can modify on an $S(k,g)$ which contains $\sqrt[4]{1/t}\, m(j,t)/n^2(n!)^2$ of the indices, in $\sqrt[4]{1/t}$ of the F_k, on $\sqrt[4]{1/t}$ of the atoms E, and improve by $\sqrt[4]{1/t}$, producing $\varepsilon(t)/2$ new errors in the uncolored partitions. This is an improvement by at least $1/t$ on colored partitions with fewer than $\varepsilon(t)$ errors on uncolored, which is a conflict. ■

Using this lemma in case ii) of Corollary 3, we can take any A, choose t so large that $n^2\sqrt[4]{1/t} < (1-A)A/4n^2(n')^2$, and then for any δ, we can get arbitrarily large j, and a δ-match, so that we can pick k, g and E where

$$\bigvee_{i\in S(k,g)} \bar{B}_i(j,t) \vee \bar{C}_i^l(j,t)/F_k \cap E \quad \text{and} \quad \bigvee_{i\in S(k,g)} \bar{B}_i'(j,t) \vee \bar{C}_i'^{l'}(j,t)/F_k \cap E$$

are $(\hat{\varepsilon}(t,A), A)$-rigidly apart, and if $(\bar{l}, \bar{l}')\in L$, then

$$\bigvee_{i\in S(k,g)} \bar{B}_i(j,t) \vee \bar{C}_i^{\bar{l}}(j,t)/F_k \cap E \quad \text{and} \quad \bigvee_{i\in S(k,g)} \bar{B}_i'(j,t) \vee \bar{C}_i'^{\bar{l}'}(j,t)/F_k \cap E$$

are $(\varepsilon(t)/2, n^2(n!)^2\sqrt[4]{1/t})$-rigidly apart. Fix t and A. Set

$$t' = \left[\inf\left(\frac{1}{A}, \frac{\sqrt[4]{t}}{n^2(n!)^2}\right)\right] \quad \text{and} \quad \bar{\varepsilon}(t') = \inf\left(\hat{\varepsilon}(t,A), \frac{\varepsilon(t)}{2}\right).$$

Now for any $(\bar{l}, \bar{l}') = L \cup \{(l, l')\}$, these two colorings of the standard modifications are $(\bar{\varepsilon}(t'), 1/t')$-rigidly apart, no matter how small δ is. This only leaves the problem of making the uncolored sequences converge in j. Noticing that δ can be made arbitrarily small, and F_k, $S(k, g)$ and E still chosen from sets of size $(1-A)A/4n^2(n!)^2$, lemmas 2 and 4 of [3] tell us that we can make our choices, for fixed t and A, hence t', with the uncolored distributions as close as we like in $\bar{d}$ to $\bigvee_{i=1}^{\operatorname{card}(S(k,g))} T^i(P)$. Thus we get the following.

LEMMA 5. *In case* ii) *of Corollary* 3, *we can construct a new array, disjointly colored in two ways, a new* $\bar{\varepsilon}(t) > 0$ *so that the set of pairs,* $\hat{L}$, $(\bar{\varepsilon}(t), 1/t)$*-rigidly apart is one larger.* ■

This construction gave us a single uncolored array. We would also like to know that we had only a single disjoint coloring, instead of two, from which our pairs are chosen. Notice that when a single distribution is disjointly colored in two different ways, the space breaks into at most $(n!)^2$ sets on each of which one disjoint coloring is the same as the other, reindexed. Our next lemma shows that rigidity is preserved if we condition on such a subset.

LEMMA 6. *If* $\bigvee_{i=1}^m B_i \vee C_i$ *and* $\bigvee_{i=1}^m B_i \vee C'_i$ *are two colored distributions* (ε, A)*-rigidly apart, then for any partition* $\{F_k\}$, *for all but* $\sqrt{A}$ *of the* F_k,

$$\bigvee_{i=1}^{m} B_i \vee C_i/F_k \quad \textit{and} \quad \bigvee_{i=1}^{m} B_i \vee C'_i/F_k$$

are still $(\varepsilon, \sqrt{A})$*-rigidly apart.*

PROOF. Same as Lemma 5. ■

Again applying lemma 2 of [3], we get a stronger version of Lemma 5.

COROLLARY 7. *In case* ii) *of Corollary* 3, *we can construct a single disjointly colored array, and set of pairs* $\hat{L}$ *one larger than L so that any pair of colorings in* $\hat{L}$, *from this single array, are* $(\hat{\varepsilon}(t), 1/t)$*-rigidly apart.* ■

With this construction in hand, we can build a single disjointly colored array, a function $\varepsilon(t) > 0$, and a set of pairs L, whose size is maximal, $(\varepsilon(t), 1/t)$-rigidly apart, and for any $(l, l') \notin L$, for any $\varepsilon_1 > 0$ if t is large enough, for any ε_2, if j is large enough

$$\bigvee_{i=1}^{m(j,t)} B_i(j,t) \vee C_i^l(j,t) \quad \text{and} \quad \bigvee_{i=1}^{m(j,t)} B_i(j,t) \vee C_i^{l'}(j,t)$$

are $(\varepsilon_2, \varepsilon_1)$-close.

Call such a "maximal arrangement."

THEOREM 1. *If in any maximal arrangement,* $\mathrm{card}(L)=0$, *then* $(T,\bar{P})$ *is v.w.b.*

PROOF. In this case, we must have, for any sequences $\bigvee_{i=1}^{m(j)} B_i(j)$ and $\bigvee_{i=1}^{m(j)} B'_i(j)$ which tend, in $\bar{d}$, to $\bigvee_{i=1}^{m(j)} T^i(P)$, as $j\to\infty$, and any monochrome colorings for these, that

$$\text{(a)}\qquad \lim_{j\to\infty} \bar{d}_{m(j)}\left(\bigvee_{i=1}^{m(j)} B_i(j)\vee C_i(j),\ \bigvee_{i=1}^{m(j)} B'_i(j)\vee C'_i(j)\right)=0.$$

Otherwise these two provide a start for our construction of a pair disjointly apart, conflicting with maximality. But (a) implies $(T,\bar{P})$ is v.w.b., as (T,P) is. ■

The other half of our argument is to show that if $L\neq\varnothing$, then $(T,\bar{P})$ is not mixing. In a maximal arrangement, notice that $\{1,\cdots,n\}$ must break into subsets $M_1,\cdots,M_s$ where $(l,l')\in L$ iff l and l' belong to different M_k's. These M_k's are our first approximation to the sets in a rotation factor for $(T,\bar{P})$. From here on many of our lemmas will be identical to those in [3]. When this is true, we will omit the proof and refer the reader to the proper lemma there. We now begin the process of selecting a maximal arrangement with $\bigvee_{i=1}^{m(j,t)} B_i(j,t)=\bigvee_{i=1}^{m(j,t)} T^i(P)$.

LEMMA 8. *If* $\bigvee_{i=1}^{m} B_i\vee C_i$ *and* $\bigvee_{i=1}^{m} B_i\vee C'_i$ *are* (ε,A)*-rigidly apart and* $S\subset\{1\cdots m\}$, *then for all but* $\sqrt{A}$ *of the atoms* $E\subset\bigvee_{i\in S^c} B_i\vee C_i\vee C'_i$,

$$\bigvee_{i\in S} B_i\vee C_i/E \quad\textit{and}\quad \bigvee_{i\in S} B_i\vee C'_i$$

are still $(\varepsilon,\sqrt{A}(m/\mathrm{card}(S^c)))$-rigidly apart.

PROOF. Same as Lemmas 4 and 6. ■

Our next result lifts lemma 7 of [3] to this case.

LEMMA 9. *In a maximal arrangement, for any* $(l,l')\notin L$ *and* $\varepsilon_1>0$, $\alpha>1$ *there is a* T, *so that for any* $t>T$, *and* $\varepsilon_2>0$ *there is a* J, δ, m *so that for* $j>J$, *and any* (α,δ,m)*-subset* $S\subset\{1,\cdots,m(j,t)\}$,

i) *for all but* ε_2 *of the atoms*

$$E\subset\bigvee_{i\in S^c} B_i(j,t)\vee\bigvee_{l=1}^{n} C_i^l(j,t),$$

$$\bar{d}_{\mathrm{card}(S)}\left(\bigvee_{i\in S} B_i(j,t)/E,\ \bigvee_{i\in S} B_i(j,t)\right)<\varepsilon_2$$

and

ii) *for all but* $2\sqrt[3]{1/t}$ *of the atoms* E,

$$\bigvee_{i\in S} B_i(j,t)\vee C_i^l(j,t)/E \quad \textit{and} \quad \bigvee_{i\in S} B_i(j,t)\vee C_i^{l'}(j,t)$$

are $(\varepsilon_2,\varepsilon_1)$*-close.*

PROOF. Statement i) follows from lemmas 2 and 4 of [3]. If ii) were false for some ε_1 and α and infinitely many t, and some ε_2, infinitely many j we could not select δ small enough or m large enough. Hence for such t and any δ, m, there are infinitely many j and (α,δ,m)-subsets $S_j\subset\{1,\cdots,m(j,t)\}$ so that for more than $2\sqrt[3]{1/t}$ of the atoms E, the pair in ii) are $(\varepsilon_2, 1-\varepsilon_1)$-rigidly apart. Using Lemma 8 and part i, we can select δ's, j's and E's so that

$$\bar d_{\mathrm{card}(S_j)}\left(\bigvee_{i\in S} B_i(j,t)/E,\ \bigvee_{i=1}^{\mathrm{card}(S_j)} T^i(P)\right)$$

is as small as we like, and for $(\bar l,\bar l')\in L$, for at least $1-\sqrt{1/t}$ of the E,

$$\bigvee_{i\in S} B_i(j,t)\vee C_i^{\bar l}/E \quad \text{and} \quad \bigvee_{i\in S} B_i(j,t)\vee C_i^{\bar l'}$$

are $(\varepsilon(t),\sqrt{1/t}(1/(1-\alpha_0)))$-rigidly apart by Lemma 6. If ii) is false we can select E, and using modifications of $\bigvee_{i\in S}B_i(j,t)/E$ and $\bigvee_{i\in S}B_i(j,t)$, build two disjointly colored arrays with set of pairs $\bar L$, of maximal size, but the falseness of ii), by the construction in Corollary 3 conflicts with maximality. ■

LEMMA 10. *If we have a maximal arrangement, then for any* ε_1 *there is a* T *so that for* $t>T$ *and any* ε_2, *there is a* δ *and a* J, *so that for* $j>J$, *and any partition* F_k *with more the* δ *its mass in sets larger than* $2^{-\delta m(j,t)}$, *then*

i) *for all but* ε_2 *of the* F_k,

$$\bar d_{m(j,t)}\left(\bigvee_{i=1}^{m(j,t)} B_i(j,t)/F_k,\ \bigvee_{i=1}^{m(j,t)} B_i(j,t)\right)<\varepsilon_2$$

and

ii) *for all but* $2\sqrt{1/t}$ *of the* F_k,

$$\bigvee_{i=1}^{m(j,t)} B_i(j,t)\vee C_i^l(j,t)/F_k \quad \textit{and} \quad \bigvee_{i=1}^{m(j,t)} B_i(j,t)\vee C_i^{l'}(j,t)$$

are $(\varepsilon_2,\varepsilon_1)$*-close, for all* $(l,l')\notin L$.

PROOF. Essentially identical to Lemma 9. ■

We now lift lemma 8 of [3] to this context.

LEMMA 11. *If we have a maximal arrangement, for any $A > 0$, there is a T, so that for $t > T$ and any $\varepsilon' > 0$, there is an M and J so that for $m > M$, $j > J$, $\varepsilon' m(j,t) \geqq m$, for all but A of the values $p = 1, \cdots, [m(j,t)/m]$, if $(l, l') \in L$ then*

$$\bigvee_{i=pm+1}^{(p+1)m} B_i(j,t) \vee C_i^l(j,t) \quad \text{and} \quad \bigvee_{i=pm+1}^{(p+1)m} B_i(j,t) \vee C_i^{l'}(j,t)$$

are $(\varepsilon(t) - 2\varepsilon', A)$-rigidly apart.

PROOF. Argue exactly as in lemma 8 of [3], using Lemma 9 instead of lemma 7 of [3], and rigid apartness instead of rigid closeness. ■

LEMMA 12. *There is a maximal arrangement with $\bigvee_{i=1}^{m(j,t)} B_i(j,t) = \bigvee_{i=1}^{m(t)+j} T^i(P)$.*

PROOF. Argue as in lemma 9 of [3], using Lemma 11 in place of Lemma 8, and Lemma 10 to get the colorings to be monochrome. ■

Call the above maximal arrangement, a perfect arrangement. We need one more basic result, the analogue of lemma 10 of [3] that a perfect arrangement is, in some sense, unique. As we noted once before, what must happen in a perfect arrangement is that for each t, the n colorings must split into subsets $M_1, \cdots, M_l$ where $(l, l') \in L$ iff l and l' belong to different M_k's.

Define the coloring $\bigvee_{i=1}^{M(t)+j} \bar{C}_i^k(M(t)+j,t)$ to be the coloring which gives each name in $\bigvee_{i=1}^{M(t)+j} T^i(P)$, all the colorings it has from any term in M_k, each with equal probability. It is these which we show are essentially unique, i.e. any two perfect arrangements differ only in how $\bigvee_{i=1}^{M(t)+j} \bar{C}_i^k(M(t)+j,t)$ is broken up into monochrome colorings.

LEMMA 13. *Suppose we have two perfect arrangements*

$$\bigvee_{i=1}^{M(t)+j} T^i(P) \vee \bigvee_{l=1}^{n} C_i^l(j,t),$$

L and $\varepsilon(t)$ and

$$\bigvee_{i=1}^{M'(t)+j} T^i(P) \vee \bigvee_{l=1}^{n} C_i'^l(j,t),$$

L' and $\varepsilon'(t)$. For any $\varepsilon > 0$, there is a T so that for $t > T$, there is a $J(t,\varepsilon)$ so that for $j > J(t,\varepsilon)$, for each k there is a k' so that for all but ε of the atoms $E \subset \bigvee_{i=1}^{j} T^i(P)$,

$$\bar{d}_j\left(\bigvee_{i=1}^{j} \bar{C}_i^k(j,t)/E, \bigvee_{i=1}^{j} \bar{C}_i'^{k'}(j,t)/E\right) \leqq \varepsilon.$$

PROOF. Assume this false for some ε_0 and infinitely many t, and for each such t, infinitely many j. For such, there must be a k so that for more than ε_0 of the E,

$$\bar{d}_j\left(\bigvee_{i=1}^{j} \bar{C}_i^k(j,t)/E, \bigvee_{i=1}^{j} \bar{C}_i^{k'}(j,t)/E\right) > \varepsilon_0 \quad \text{for all } k'.$$

There must, then, be $(l,l') \in L'$ and $(\bar{l},\bar{l}') \notin L$ and sets $F_1, F_2 \subset \bigvee_{i=1}^{j} T^i(P)$ with $\mu(F_1) = \mu(F_2) \geqq \varepsilon_0/n^4$ so that

$$\bigvee_{i=1}^{j} T^i(P) \vee C_i^{l}(j,t)/F_1 = \bigvee_{i=1}^{j} T^i(P) \vee C_i^{\bar{l}}(j,t)/F_1$$

and

$$\bigvee_{i=1}^{j} T^i(P) \vee C_i^{l'}(j,t)/F_2 = \bigvee_{i=1}^{j} T^i(P) \vee C_i^{\bar{l}'}(j,t)/F_2$$

for infinitely many j. But the right hand sides are $(\varepsilon_2, \varepsilon_1)$-close, $\varepsilon_2 \underset{j}{\to} 0$, $\varepsilon_1 \underset{t}{\to} 0$, by Lemma 10, where the left hand sides are still $(\varepsilon'(t), \sqrt{1/t})$-rigidly apart by Lemma 6. This is a conflict. ∎

This, now, gives us our last setup.

COROLLARY 14. *For any $\varepsilon > 0$, there is a T_0,*
so that for all $t_0 > T_0$, there is a J_0,
so that for all $j_0 > J_0$, there is a T,
so that for all $t > T$, there is a J,
so that for all $j > J$, for all but ε of the $p \in 1, \cdots, [m(j_1, t_1)/m(j_0, t_0)]$, for all k there is a k' so that

$$\bar{d}_{m(j_0,t_0)}\left(\bigvee_{i=1}^{m(j_0,t_0)} T^i(P) \vee \bar{C}_i^k(j_0,t_0), \bigvee_{pm(j_0,t_0)+1}^{(p+1)m(j_0,t_0)} T^i(P) \vee \bar{C}_i^{k'}(j,t)\right) < \varepsilon.$$

PROOF. Applying Lemma 11, our choices can be made so that for all but ε of the values p, $\bigvee_{pm(j_0,t_0)+1}^{(p+1)m(j_0,t_0)} T^i(P) \vee \bar{C}_i^{k'}(j,t)$ comes from a lumping of an M_k^i from a perfect arrangement. Lemma 13 now implies the result. ∎

Now define partitions $\{H_1(j,t), \cdots, H_s(j,t)\}$ where $\omega \in H_k(s,t)$ iff the $T, \bar{P}$-name of ω is a name in $\bigvee_{i=1}^{M(t)+j} \bar{C}_i^k(j,t)$. Theorem 1 says that if $s = 1$, then $(T, \bar{P})$ is Bernoulli. We now get the other half of our result. ∎

THEOREM 2. *If $s > 1$, then $(T, \bar{P})$ is not mixing.*

PROOF. By Corollary 13, for any ε there is a T_0 so that for $t_0 > T_0$, there is a J_0 so that for $j_0 > J_0$, there are arbitrarily large values k with

$$T^k(H_k(j_0, t_0)) \cap H_{k'}(j_0, t_0) \begin{cases} < \varepsilon & \text{or} \\ > 1 - \varepsilon & \text{for any } k, k' \leqq s. \end{cases}$$

Thus $(T, \bar{P})$ is not mixing. ■

COROLLARY 15. *If $(T, \bar{P})$ is not Bernoulli, then it has a rotation factor.*

PROOF. Follows now from Lemma 1. ■

This completes our result. There are a number of directions in which to pursue this kind of technique. We can try to extend to as general a group extension as possible. The new difficulty that arises is that the skewing function g may not be finite valued. B. Weiss has already shown how to handle a similar situation in [5]. Ornstein has introduced the notion of two factors sitting the same in a Bernoulli shift [1]. We now can build many different factors with finite fibers in a Bernoulli shift. Do they all sit the same? How many factors with k-point fibers are there? The answer is, there are only finitely many. This argument will appear separately. A third area with real promise is to "relativize" these arguments, to finite extensions of a direct product of a Bernoulli shift and some other transformation, and show, through the relativized isomorphism theory [4], that either this other transformation still has a Bernoulli complement, or the extension can be made measurable with respect to it. This program almost certainly will work.

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